Appl. Math. Lett. Vol. 3, No. 2, pp. 21–23, 1990
0893-9659/90 $3.00 + 0.00

Complete Systems of Functions for the Exterior Dirichlet and Neumann Problems in the Bending of Mindlin-Type Plates

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Abstract. Complete systems of functions in the space of the solution are found for the exterior Dirichlet and Neumann problems in a theory of bending of plates with transverse shear deformation, which allow the application of the method of generalized Fourier series.

Let $S \subset \mathbf{R}^2$ be a finite domain bounded by a simple closed C^2-curve ∂S, $n = (n_1, n_2)^T$ the unit outward normal to ∂S (the superscript T indicates matrix transposition), and $x = (x_1, x_2)$ a generic point in $\mathbf{R}^2$. In [1] and [2], a two-dimensional theory of bending of plates with transverse shear deformation is investigated, which reduces to the equilibrium equations

$$L(\partial_x)v(x) = 0, \quad x \in S, \tag{1}$$

and boundary conditions of the form $v(x) = A(x)$, $x \in \partial S$ (Dirichlet), or $T(\partial_x)v(x) = B(x)$, $x \in \partial S$ (Neumann). Here $v = (v_1, v_2, v_3)^T$ is a vector characterizing the displacements, $L(\partial_x) = L(\partial/\partial x_1, \partial/\partial x_2)$,

$$\begin{aligned} L(\xi) &= L(\xi_1, \xi_2) \\ &= \begin{pmatrix} h^2\mu\Delta + h^2(\lambda+\mu)\xi_1^2 - \mu & h^2(\lambda+\mu)\xi_1\xi_2 & -\mu\xi_1 \\ h^2(\lambda+\mu)\xi_1\xi_2 & h^2\mu\Delta + h^2(\lambda+\mu)\xi_2^2 - \mu & -\mu\xi_2 \\ \mu\xi_1 & \mu\xi_2 & \mu\Delta \end{pmatrix}, \end{aligned}$$

$\Delta = \xi_1^2 + \xi_2^2$, $T(\partial_x) = T(\partial/\partial x_1, \partial/\partial x_2)$ is the boundary stress operator defined by the matrix

$$\begin{pmatrix} h^2(\lambda+2\mu)n_1\xi_1 + h^2\mu n_2\xi_2 & h^2\mu n_2\xi_1 + h^2\lambda n_1\xi_2 & 0 \\ h^2\lambda n_2\xi_1 + h^2\mu n_1\xi_2 & h^2\mu n_1\xi_1 + h^2(\lambda+2\mu)n_2\xi_2 & 0 \\ \mu n_1 & \mu n_2 & \mu(n_1\xi_1 + n_2\xi_2) \end{pmatrix},$$

$\sqrt{12}\,h$ is the constant thickness of the plate, λ and μ are the Lamé coefficients of the material, and $A(x)$ and $B(x)$ vectors prescribed on ∂S.

The results of [1] and [2] show that, since $v = O(|x|^2 \ln|x|)$ as $|x| = (x_1^2 + x_2^2)^{\frac{1}{2}} \to \infty$, the solution of the interior and exterior Dirichlet and Neumann problems by means of the boundary integral equation method requires the introduction of two special uniqueness classes $\mathcal{A}$ and $\mathcal{A}^*$. Although existence theorems in these classes have been established, various other procedures do not become routinely applicable in the case of bending of plates because of the growth of the corresponding solutions at infinity. The list of casualties includes Kupradze's technique of generalized Fourier series [3], based on the construction of a complete set in the space of the solution. Our intention in what follows is to show that, despite the handicap mentioned above, such sets can still be obtained for the exterior problems of bending, which allow this useful approximation method to remain operative.

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We adopt the notation $S_+ = S$ and $S_- = \mathbf{R}^2 \setminus \bar{S}$ and extend it to any finite two-dimensional domain G whose boundary ∂G is a simple closed contour. For $\varphi \in C(S_+)$ we denote by $\varphi^+(x)$ the limiting value (if it exists) of $\varphi(x')$ as $S_+ \ni x' \to x \in \partial S$, and by $\varphi^-(x)$ the similar limit for $x' \in S_-$. If $\varphi = (\varphi_1, \varphi_2, \varphi_3)^T$ and all the φ_i belong to the same space X, we also write $\varphi \in X$, and if Φ is a mapping on X, then $\Phi\varphi = (\Phi\varphi_1, \Phi\varphi_2, \Phi\varphi_3)^T$. Finally, we denote the columns of a matrix M by $M^{(i)}$.

THEOREM 1. *Let G be a finite domain in $\mathbf{R}^2$, X a space of functions defined on ∂G, Φ a linear functional on X, and*

$$u(x) = \Phi_y\big(D(x, y)\varphi(y)\big), \quad \varphi \in X, \quad x \in G_-,$$

where D is a fundamental matrix of solutions for (1) *[2] and the subscript y means that Φ operates with respect to y. Then $u \in \mathcal{A}$ if and only if*

$$\begin{aligned} &\Phi_y\big(\varphi_\alpha(y) - y_\alpha\varphi_3(y)\big) = 0, \quad \alpha = 1, 2, \\ &\Phi_y\varphi_3(y) = 0. \end{aligned} \tag{2}$$

The proof of this assertion is based on the behaviour of D for y fixed on ∂G and $|x|$ large.

REMARK 1. *Clearly, the matrix of fundamental solutions can be represented in the form*

$$D(x, y) = D_\infty(x, y) + \tilde{D}(x, y), \tag{3}$$

where $D_\infty(x, y)$ and $\tilde{D}(x, y)$ are such that $\Phi_y\big(D_\infty(x, y)\varphi(y)\big) = 0$ when (2) hold, and $\tilde{u}(x) = \Phi_y\big(\tilde{D}(x, y)\varphi(y)\big)$.

THEOREM 2. *$D^{(i)}(x, y)$ and $\tilde{D}^{(i)}(x, y)$, $i = 1, 2, 3$, are solutions of* (1) *in S_- for any (fixed) $y \in \partial S$. In addition, $\tilde{D}^{(i)} \in \mathcal{A}$.*

This assertion follows from the comparison of the asymptotics of these functions with the general far-field pattern established in [4].

Let S' be a finite domain in $\mathbf{R}^2$ with a simple closed boundary $\partial S'$ that is sufficiently smooth to allow the application of the divergence theorem, and suppose that $\bar{S}'$ lies strictly in S_+ and that the origin lies in S'_+. Also, let $\{x^{(k)}\}_{k=1}^\infty$ be a countable set of points densely distributed on $\partial S'$.

THEOREM 3. *The set*

$$\{\psi^{(i)}, \theta^{(i,k)}\}, \quad i = 1, 2, 3, \quad k = 1, 2, \dots, \tag{4}$$

where $\psi^{(1)}(x) = (1, 0, -x_1)^T$, $\psi^{(2)}(x) = (0, 1, -x_2)^T$, $\psi^{(3)}(x) = (0, 0, 1)^T$, and $\theta^{(i,k)}(x) = D^{(i)}(x, x^{(k)})$, is complete in $L^2(\partial S)$.

PROOF: Suppose that there are constants c_i and c_{ik}, $i = 1, 2, 3$, $k = 1, \dots, N$, not all zero, such that

$$w(x) = \sum_{i=1}^{3}\sum_{k=1}^{N} c_{ik}\theta^{(i,k)}(x) + \sum_{i=1}^{3} c_i\psi^{(i)}(x) = 0, \quad x \in \partial S. \tag{5}$$

Then $Tw = 0$ on ∂S. Writing $w = w_\infty + \tilde{w} + \sum_{i=1}^3 c_i\psi^{(i)}$, where w_∞ and $\tilde{w}$ are defined by (5) in terms of $D_\infty^{(i)}$ and $\tilde{D}^{(i)}$, respectively, we deduce from Theorem 2 that $L\tilde{w} = 0$ in S_-, $T\tilde{w} = -Tw_\infty$ on ∂S, and $\tilde{w} \in \mathcal{A}$. By the solvabilty condition for the exterior Neumann problem [1], $\int\limits_{\partial S} (\psi^{(i)})^T Tw_\infty \, ds = 0$, $i = 1, 2, 3$.

Let K_R be a disk with centre at the origin and radius R such that $\bar{S}_+ \subset K_R$ strictly. Applying the reciprocity relation [2] to $\psi^{(i)}$ and w_∞ in $K_R \setminus \bar{S}_+$, we find that $\int_{\partial K_R}$

$(\psi^{(i)})^T T w_\infty \, ds = 0$, $i = 1, 2, 3$. A lengthy but straightforward calculation shows that this is equivalent to

$$\sum_{k=1}^{N} (c_{\alpha k} - x_\alpha^{(k)} c_{3k}) = 0, \quad \alpha = 1, 2, \quad \sum_{k=1}^{N} c_{3k} = 0. \tag{6}$$

Let X be the space of bounded functions on $\partial S'$, Φ the linear functional on X defined by $\Phi\varphi = \sum_{k=1}^{N} \varphi(x^{(k)})$, $\varphi \in X$, and $\varphi_c \in X$ a vector such that $\varphi_c(x^{(k)}) = (c_{1k}, c_{2k}, c_{3k})^T$, $k = 1, 2, \ldots, N$. Then $\sum_{i=1}^{3} \sum_{k=1}^{N} c_{ik} \theta^{(i,k)}(x) = \Phi_y\big(D(x, y)\varphi_c(y)\big)$. On the other hand, (6) are the equivalent of (2), consequently, by Theorem 1 and the definition of $\mathcal{A}^*$ [1], $w \in \mathcal{A}^*$.

In view of (5), we also have $Lw = 0$ in S_- and $w = 0$ on ∂S. By the uniqueness theorem for the exterior Dirichlet problem [1], $w = 0$ in $\bar{S}_-$. Analyticity arguments now imply that $w = 0$ in S'_-, which is obviously impossible. Therefore, (5) holds only if $c_{ik} = c_i = 0$, $i = 1, 2, 3$, $k = 1, \ldots, N$. This proves the linear independence of the set (4) on ∂S.

Next, let $\varphi \in L^2(\partial S)$ be such that

$$\int_{\partial S} (\theta^{(i,k)})^T \varphi \, ds = \int_{\partial S} (\psi^{(i)})^T \varphi \, ds = 0, \quad i = 1, 2, 3, \ k = 1, 2, \ldots \tag{7}$$

The elastic single layer potential [1] $V(x) = \int_{\partial S} D(x, y)\varphi(y) \, ds(y)$ satisfies $LV = 0$ in S'_+ and $V = 0$ on $\partial S'$, the latter in view of (7) and the fact that the $x^{(k)}$ are densely distributed on $\partial S'$. Hence [2], $V = 0$ in $\bar{S}'_+$. By analyticity, $V = 0$ in S_+, which implies that $(TV)^+ = 0$. Considering a particular case when $S_+ \ni x' \to x \in \partial S$ along the normal to ∂S at x, we deduce [5] that $\varphi \in C^{0,\alpha}$ for any $\alpha \in (0, 1)$. Then $V \in C(\mathbb{R}^2)$, consequently, $V = 0$ on ∂S.

Let $X = C(\partial S)$, and let Φ this time be the linear functional on X defined by $\Phi\varphi = \int_{\partial S} \varphi \, ds$. The second set of equalities in (7) are the equivalent of (2), therefore, by Theorem 1, $V \in \mathcal{A}$. By the uniqueness theorem for the exterior Dirichlet problem [2], $V = 0$ in S_-, which also yields $(TV)^- = 0$. Using the explicit formulae for $(TV)^\pm$ in the case of a $C^{0,\alpha}$-density [1], we now conclude that $\varphi = 0$.

The completeness of the set (4) follows from the fact that $L^2(\partial S)$ is a Hilbert space.

REMARK 2. *In classical elasticity [3] the $\psi^{(i)}$ do not need to be included in (4).*

THEOREM 4. *The set*

$$\{\nu^{(i,k)}\}, \quad i = 1, 2, 3, \ k = 1, 2, \ldots, \tag{8}$$

where $\nu^{(i,k)}(x) = T(\partial_x) D^{(i)}(x, x^{(k)})$, is complete in $L^2(\partial S)$.

The proof of this assertion is similar to that of Theorem 3.

The generalized Fourier series method of approximate solution of the exterior Dirichlet and Neumann problems for (1) now proceeds as in three-dimensional elasticity [3], with the use of the set (4) for the former, and of (8) for the latter.

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